Pergamon

Int. J. Solids Structures Vol. 34, Nos 33–34, pp. 4399–4419, 1997

0020–7683/97 $17.00 + .00

PII : S0020–7683(97)00019–X

NONLOCAL STRAIN SOFTENING BAR REVISITED

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(*Received* 14 *April* 1996, *in revised form* 6 *December* 1996)

Abstract—Localization and dissipation in a nonlocal strain softening bar are investigated analytically. A nonlocal elastic–plastic model consistent with thermodynamic theory is proposed. The flow rules of nonlocal associated plasticity are derived by invoking a generalized form of the principle of maximum dissipation in classical plasticity. Analytical solutions are derived for a linearly strain softening material. The relationship between nonlocal characteristic length and width of localized zone is elucidated. A comparison with results obtained from corresponding gradient approach is presented. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Localized failure in a rate-independent material is associated with material instability. Mathematically, in static or quasi-static problems, the onset of localization results in loss of ellipticity of the governing partial differential equations. Accordingly, post-localized phenomena cannot be described within the framework of classical continua. Considerable interest has been directed at theories that treat localization as a bifurcation from a state of homogeneous deformation. Such an approach—see Miehe and Schröder (1994), for example, for a recent survey—is not adopted in the present paper, however, but strain fields are assumed to remain continuous everywhere.

From a computational point of view, the loss of ellipticity leads to numerical instability and mesh sensitivity. If conventional continuum models are applied, finite-element solutions show non-objectivity with respect to the mesh for standard finite elements. As a consequence, the localized zone shrinks into a region of vanishing volume under zero energy dissipation when the elements tend to become infinitely small. Objectively can be achieved by different approaches, e.g., by one involving the concept of localization limiters that force the localized zone to have a certain minimum finite size (see Belytschko and Lasry (1989) for a survey). Cohesive crack models represent simple examples of the use of localization limiters. Such models are used widely by researchers within the field of fracture mechanics; for a recent review, see Elices *et al.* (1993).

A general form of localization limiters can be utilized within the theory of nonlocal continua. Nonlocal models for localized failure based on theories of either plasticity or damage (or a combination of both) have been employed rather frequently during the past decade. Plasticity-based models have been proposed by Bažant and Feng-Bao Lin (1988) and Nilsson (1994). Damage-based models have been proposed by Bažant and Pijaudier-Cabot (1988), Bažant and Ožbolt (1990), Murakami *et al.* (1993), and—based on bifurcation analysis—by Leblond *et al.* (1994) and Pijaudier-Cabot and Benallal (1993). (Contributions of several other authors could be mentioned as well; the list is in no way complete.)

In addition to the nonlocal continuum approach various other continuum models are applicable to describing localized failure. Rate-dependent models have been proposed to eliminate problems due to change of type of differential equations (from hyperbolic to parabolic or elliptic in the case of strain softening materials), e.g., by Sandler and Wright (1984), Needleman (1988), Loret and Prevost (1990) and Sluys and de Borst (1992). Localized failure has likewise been investigated within the framework of continuum models, both for materials with gradient effects (recently by Mühlhaus and Aifantis (1991), de Borst and Mühlhaus (1992), Fleck *et al.* (1994) and many others) and for Cosserat media (e.g., by Mülhaus and Vardoulakis (1987), de Borst (1991) and Steinmann (1995)). One should also note that various micromechanics models have been proposed for the analysis of localized failure in different types of heterogeneous solids and composite materials; for a review, see e.g., Nemat-Nasser and Hori (1993).

One-dimensional strain softening has been analyzed by a large number of authors, using the one approach or the other of those just referred to. Indeed, many strain softening models intended for three-dimensional problems probably represent extensions to three dimensions of a physical understanding and of mathematical representations applying originally to one dimension. Although such extensions generally cannot be considered unambiguous, it surely can be argued that one-dimensional considerations usually suffice for exposing the essential features of the phenomena that are expected to be predicted by the theory in question. Also, needless to say, analytical solutions have with few exceptions been found only for one-dimensional problems.

A number of authors have used a gradient approach for finding analytical solutions to the problem of localization in solids, among these Aifantis (1984), Coleman and Hodgon (1985) and Triantafyllides and Aifantis (1986), and more recently Schreyer (1990) and de Borst and Mühlhaus (1992). Due to the complexity of the corresponding nonlocal approach, the issue of finding analytical solutions to problems of localization is not an easy one. Among those who have endeavoured to find such solutions are Bažant and Zubelewic (1988), Valanis (1991) and Brekelman (1993) (the work of each of them being based on damage theory).

This paper is organized as follows. In Section 2 a nonlocal model consistent with thermodynamic theory is formulated and the flow rules of nonlocal associated plasticity are derived from a generalization of the principle of maximum dissipation. In Section 3 a strain softening bar loaded in uniaxial tension is investigated analytically. An integral equation from which the width of the localized zone can be obtained is derived. Also, the role of characteristic lengths of a nonlocal continuum is discussed. In Section 4, finally, the question of uniqueness of the solutions is addressed, a comparison is made with gradient theory, and various concluding comments are offered.

2. NONLOCAL PLASTICITY

Classical continuum mechanics is based on the principle of local action and on the assumption that the equations of balance are valid for every part of a given body, however small it may be. The principle of local action is not valid in nonlocal theories, however. For example, stress is affected not only by an infinitesimally small region around the actual stress point, but by the entire body in question. Continuum theories including such long-range interactions have been in the focus of interest now for over twenty years; see e.g., Kunin (1982, 1983), Gurtin and Williams (1971) and the review by Edelen (1976). Common to nonlocal theories is the postulation of balance equations for a given body in its entirety (global equations) and not for just an arbitrary part of it. Corresponding local equations are only valid then after the incorporation of nonlocal residuals which account for the long-range interactions. There is no unique approach, however, to the problem of describing nonlocal interactions in continuous media. An entirely different approach to nonlocality is one based on micromechanics of heterogeneous materials, according to which the nonlocal effects manifest themselves when the appropriate classical field equations are solved; see Hashin and Shtrikman (1962a, b), Willis (1981), Nemat-Nasser and Hori (1993), and the nonlocal formulation concerning elastic composites presented recently by Drugan and Willis (1996). Theories of nonlocal plasticity have been developed by Eringen (1981, 1983), Nilsson (1994) and—based on crystal plasticity—by Berveiller *et al.* (1993). The nonlocal approach employed in the present paper is similar to that of Edelen (1976).

Nonlocal constitutive variables will be constructed here from a set of basic state functions, constituted by total strain, plastic strain and a scalar measure of strain hardening (Section 2.1). A rate-independent model consistent with thermodynamic theory, in which stress and free energy are assumed to be functions of the nonlocal variables, will then be formulated (Section 2.2).

A yield function, in which the same set of nonlocal independent variables occurs as in the case of the response functions of stress and free energy will be introduced. This is fundamental for the theory. In classical plasticity the yield condition implies that whether a state is elastic or plastic depends only on the inelastic state at the actual stress point and not on the state at neighbouring points, hence excluding any dependence on the gradients of the inelastic variables. From a physical standpoint, however, it is difficult to find support for rejecting, as Eringen (1981) did, long-range interactions in the yield criterion in nonlocal plasticity; cf. the paper by Kratochvil (1988), in which nonlocality and the microstructural origin of plastic deformation are discussed.

Statements of the Second Law of thermodynamics (such as the Clausius–Duhem inequality) are often replaced in local and purely mechanical plasticity theory by a work assumption of some kind, such as the postulate of Drucker (1952) or that of Il'iushin (1961). Drucker's postulate, concerning the non-negativity of work in a cycle of stress, is a stability condition that is valid for hardening materials only, whereas the restrictions that the invoking of Il'iushin's postulate, which concerns a strain cycle, place on the constitutive equations are valid for both hardening and softening behaviour. From the postulate of either Drucker or Il'iushin, each formulated within the context of linearized theory and small deformations, conditions for normality of plastic strain rate and for convexity of the yield surfaces can be obtained. In nonlocal plasticity it is not possible to derive such conditions from any generalized postulates of the Drucker or Il'iushin type. However, it will be shown that it is in fact possible to derive the flow rules of nonlocal associated plasticity by invoking a generalized form of the principle of maximum dissipation in classical plasticity (Section 2.2.2).

2.1. *Attenuation functions*

The basic *functions* (state functions) that are assumed to constitute the elastic–plastic state are represented by $\boldsymbol{\varepsilon}(\mathbf{x}, t)$, $\boldsymbol{\varepsilon}^p(\mathbf{x}, t)$ and $\kappa(\mathbf{x}, t)$, which represent total strain, plastic strain and strain hardening, respectively, as was discussed previously.† Thus, one can see that, in contrast to local theory, these functions—and not simply their values at $\mathbf{x}$—are required for specifying the dependent variables. In the case of Helmholtz free energy ψ, this means that the value of ψ at $\mathbf{x}$ is determined by the values of the state functions over the entire body. In order to provide for such dependence the following quantities are constructed:

$$\left.\begin{aligned}\langle\boldsymbol{\varepsilon}\rangle(\mathbf{x},t) &= \frac{1}{V_e(\mathbf{x})}\int_B w^e(|\mathbf{z}-\mathbf{x}|)\boldsymbol{\varepsilon}(\mathbf{z},t)\,\mathrm{d}V(\mathbf{z}),\\ \langle\boldsymbol{\varepsilon}^p\rangle(\mathbf{x},t) &= \frac{1}{V_p(\mathbf{x})}\int_B w^p(|\mathbf{z}-\mathbf{x}|)\boldsymbol{\varepsilon}^p(\mathbf{z},t)\,\mathrm{d}V(\mathbf{z}),\\ \langle\kappa\rangle(\mathbf{x},t) &= \frac{1}{V_h(\mathbf{x})}\int_B w^h(|\mathbf{z}-\mathbf{x}|)\kappa(\mathbf{z},t)\,\mathrm{d}V(\mathbf{z}),\end{aligned}\right\} \tag{1}$$

where V_i, $i = (e, p, h)$ are defined by

$$V_i(\mathbf{x}) = \int_B w^i(|\mathbf{z}-\mathbf{x}|)\,\mathrm{d}V(\mathbf{z}). \tag{2}$$

Here w^e, w^p and w^h are scalar, time independent *attenuation* or *influence functions*, depending on position as indicated, by which *representative volumes* V_e, V_p and V_h, that are characteristic measures of body B of volume $V(B)$, are constructed. On the basis of physical considerations, it is reasonable to assume that the attenuation functions decay smoothly

† Standard vector and tensor notations are employed, the bold letters denoting vectors and second order tensors.

and rapidly with increasing distance from $\mathbf{x}$, as is the case when w^e, w^p and w^h are assumed to be of exponential form. It is evident that the nonlocal feature of a given material behaviour is affected considerably by the choice of attenuation functions. It is assumed that the attenuation functions are normalized, $w^e(0) = w^p(0) = w^h(0) = 1$. For the moment, it is not necessary to specify the attenuation functions further. Before continuing, it is convenient to introduce some new notations. Define the functions $\tilde{w}^i(\mathbf{z}, \mathbf{x})$ and $\tilde{w}^i(\mathbf{x}, \mathbf{z})$, using the relationship

$$\tilde{w}^i(\mathbf{z}, \mathbf{x}) = \frac{1}{V(\mathbf{x})} w^i(|\mathbf{z}-\mathbf{x}|) \tag{3}$$

and

$$\tilde{w}^i(\mathbf{x}, \mathbf{z}) = \frac{1}{V(\mathbf{z})} w^i(|\mathbf{x}-\mathbf{z}|), \tag{4}$$

respectively, where i stands as before for either e, p or h. *Braces* are used to denote the averaging operator

$$\overline{\{Q\}}_i = \int_B \tilde{w}^i(\mathbf{x}, \mathbf{z}) Q(\mathbf{z})\, \mathrm{d}V(\mathbf{z}), \tag{5}$$

valid for every scalar, vector or tensor valued function Q.

The function $\int_B \tilde{w}^i(\mathbf{x}, \mathbf{z})\, \mathrm{d}V(\mathbf{z})$ is denoted simply as β^i, i.e.

$$\beta^i = \int_B \tilde{w}^i(\mathbf{x}, \mathbf{z})\, \mathrm{d}V(\mathbf{z}) = \overline{\{1\}}_i, \tag{6}$$

where the second equality follows from (5).

From (2) and (3) it follows that $\tilde{w}^i$ satisfies the normalization condition

$$\int_B \tilde{w}^i(\mathbf{z}, \mathbf{x})\, \mathrm{d}V(\mathbf{z}) = 1, \tag{7}$$

and that (1) can be replaced by

$$\left.\begin{aligned} \langle \boldsymbol{\varepsilon} \rangle(\mathbf{x}, t) &= \int_B \tilde{w}^e(\mathbf{z}, \mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{z}, t)\, \mathrm{d}V(\mathbf{z}), \\ \langle \boldsymbol{\varepsilon}^p \rangle(\mathbf{x}, t) &= \int_B \tilde{w}^p(\mathbf{z}, \mathbf{x}) \boldsymbol{\varepsilon}^p(\mathbf{z}, t)\, \mathrm{d}V(\mathbf{z}), \\ \langle \kappa \rangle(\mathbf{x}, t) &= \int_B \tilde{w}^h(\mathbf{z}, \mathbf{x}) \kappa(\mathbf{z}, t)\, \mathrm{d}V(\mathbf{z}). \end{aligned}\right\} \tag{8}$$

Remarks 2.1. Here $\mathbf{z}$ is used to indicate functional relationship in the sense that $\mathbf{z}$ represents all points in the body, whereas $\mathbf{x}$ represents an arbitrarily distinguished point. When no confusion can result, the dependence of the arguments on $\mathbf{x}$ and t is omitted.

It should be emphasized that $\langle \cdot \rangle$ is a function symbol not to be confused with the notion of averaging a function.

The quantities $\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}^p$ and κ are referred to as *local* variables, whereas the corresponding quantities $\langle \boldsymbol{\varepsilon} \rangle$, $\langle \boldsymbol{\varepsilon}^p \rangle$ and $\langle \kappa \rangle$ are referred to as *nonlocal* variables. Thus, for example $\boldsymbol{\varepsilon}^p$ is local plastic strain and $\langle \boldsymbol{\varepsilon}^p \rangle$ is nonlocal plastic strain.

Note that if one takes $w = \delta$ (the Dirac delta function) in (1), it follows from (2) that $V(x) = 1$, and hence that the original state function is recovered.

Note also that, due to the normalization condition (7), nonlocal quantities become identical with corresponding local ones during homogeneous motion. □

2.2. *Thermodynamic basis*

Assume now that

$$\left.\begin{aligned}\boldsymbol{\sigma} &= \tilde{\boldsymbol{\sigma}}(\langle\boldsymbol{\varepsilon}\rangle, \langle\boldsymbol{\varepsilon}^p\rangle, \langle\kappa\rangle),\\ \psi &= \tilde{\psi}(\langle\boldsymbol{\varepsilon}\rangle, \langle\boldsymbol{\varepsilon}^p\rangle, \langle\kappa\rangle).\end{aligned}\right\} \tag{9}$$

In a purely mechanical theory the second law of nonlocal thermodynamics reduces to

$$\int_B (-\dot{\psi} + \boldsymbol{\sigma}\cdot\dot{\boldsymbol{\varepsilon}})\,\mathrm{d}V \geqslant 0, \tag{10}$$

the superposed dots denoting material time derivatives. Note that, because of (8) and $(9)_2$, the time derivative of the Helmholtz free energy can be expressed in the form

$$\dot{\psi} = \frac{\partial\tilde{\psi}}{\partial\langle\boldsymbol{\varepsilon}\rangle}\cdot\int_B \tilde{w}^e(\mathbf{z},\mathbf{x})\dot{\boldsymbol{\varepsilon}}(\mathbf{z})\,\mathrm{d}V(\mathbf{z}) + \frac{\partial\tilde{\psi}}{\partial\langle\boldsymbol{\varepsilon}^p\rangle}\cdot\int_B \tilde{w}^p(\mathbf{z},\mathbf{x})\dot{\boldsymbol{\varepsilon}}^p(\mathbf{z})\,\mathrm{d}V(\mathbf{z})$$
$$+\frac{\partial\tilde{\psi}}{\partial\langle\kappa\rangle}\int_B \tilde{w}^h(\mathbf{z},\mathbf{x})\dot{\kappa}(\mathbf{z})\,\mathrm{d}V(\mathbf{z}), \tag{11}$$

allowing the second law (10) to be written as

$$-\int_B\left[\int_B \tilde{w}^e(\mathbf{z},\mathbf{x})\frac{\partial\tilde{\psi}}{\partial\langle\boldsymbol{\varepsilon}\rangle}(\mathbf{x})\cdot\dot{\boldsymbol{\varepsilon}}(\mathbf{z})\,\mathrm{d}V(\mathbf{z}) + \int_B \tilde{w}^p(\mathbf{z},\mathbf{x})\frac{\partial\tilde{\psi}}{\partial\langle\boldsymbol{\varepsilon}^p\rangle}(\mathbf{x})\cdot\dot{\boldsymbol{\varepsilon}}^p(\mathbf{z})\,\mathrm{d}V(\mathbf{z})\right.$$
$$\left.+\int_B \tilde{w}^h(\mathbf{z},\mathbf{x})\frac{\partial\tilde{\psi}}{\partial\langle\kappa\rangle}(\mathbf{x})\dot{\kappa}(\mathbf{z})\,\mathrm{d}V(\mathbf{z})\right]\mathrm{d}V(\mathbf{x}) + \int_B \boldsymbol{\sigma}(\mathbf{x})\cdot\dot{\boldsymbol{\varepsilon}}(\mathbf{x})\,\mathrm{d}V(\mathbf{x}) \geqslant 0. \tag{12}$$

Observe that interchanging $\mathbf{x}$ and $\mathbf{z}$ and reversing the order of integration in (12) allows, in view of (5), the second law to be cast in the form

$$\int_B\left[\left(\boldsymbol{\sigma} - \overline{\left\{\frac{\partial\tilde{\psi}}{\partial\langle\boldsymbol{\varepsilon}\rangle}\right\}_e}\right)\cdot\dot{\boldsymbol{\varepsilon}} - \overline{\left\{\frac{\partial\tilde{\psi}}{\partial\langle\boldsymbol{\varepsilon}^p\rangle}\right\}_p}\cdot\dot{\boldsymbol{\varepsilon}}^p - \overline{\left\{\frac{\partial\tilde{\psi}}{\partial\langle\kappa\rangle}\right\}_h}\dot{\kappa}\right]\mathrm{d}V \geqslant 0. \tag{13}$$

The inequality (13) must hold during both loading and unloading. Since unloading corresponds to vanishing $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\kappa}$ for any value of $\boldsymbol{\varepsilon}$ inside some bounding surface (the elastic region in strain space), and since the inequality is linear in $\dot{\boldsymbol{\varepsilon}}$, it follows that

$$\boldsymbol{\sigma} = \overline{\left\{\frac{\partial\tilde{\psi}}{\partial\langle\boldsymbol{\varepsilon}\rangle}\right\}_e}, \tag{14}$$

during unloading (or neutral loading). Note that $\boldsymbol{\sigma}$ is evaluated at fixed but arbitrary values of the inelastic variables. Accordingly, (14) is valid for every $\boldsymbol{\varepsilon}^p$ and κ. Hence, since $\boldsymbol{\sigma}$ is independent of $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\kappa}$, the result given by (14) remains valid during loading as well. Substituting (14) back into (13) then yields

$$\int_B \left[-\overline{\left\{ \frac{\partial \tilde{\psi}}{\partial \langle \boldsymbol{\varepsilon}^p \rangle} \right\}_p} \cdot \dot{\boldsymbol{\varepsilon}}^p - \overline{\left\{ \frac{\partial \tilde{\psi}}{\partial \langle \kappa \rangle} \right\}_h} \dot{\kappa} \right] \mathrm{d}V \geqslant 0, \tag{15}$$

the left hand side of the inequality being an expression of the total plastic dissipation $\mathbb{D}^p$ in the body, i.e.

$$\mathbb{D}^p = \int_B \left[-\overline{\left\{ \frac{\partial \tilde{\psi}}{\partial \langle \boldsymbol{\varepsilon}^p \rangle} \right\}_p} \cdot \dot{\boldsymbol{\varepsilon}}^p - \overline{\left\{ \frac{\partial \tilde{\psi}}{\partial \langle \kappa \rangle} \right\}_h} \dot{\kappa} \right] \mathrm{d}V$$
$$= \int_B \left[-\frac{\partial \tilde{\psi}}{\partial \langle \boldsymbol{\varepsilon}^p \rangle} \cdot \langle \dot{\boldsymbol{\varepsilon}}^p \rangle - \frac{\partial \tilde{\psi}}{\partial \langle \kappa \rangle} \langle \dot{\kappa} \rangle \right] \mathrm{d}V, \tag{16}$$

where the second equality is a trivial consequence of (12) and (13).

Remarks 2.2. It is not difficult to formulate the theory for elastic–plastic bodies undergoing finite deformation. However, to avoid unnecessary complications, here $\boldsymbol{\varepsilon}$ can be identified as infinitesimal strain and $\boldsymbol{\sigma}$ as actual (Cauchy) stress. Accordingly, $\mathbf{x}$ and $\mathbf{z}$ can both be interpreted as the position of a material point in some reference configuration. □

2.2.1. *Restricted nonlocality*. In many cases, it is essential that stress rather than strain be used as an independent state function. In general, it appears more or less impossible to invert the stress–strain relation $(9)_1$ in order to establish the strain at the actual stress point as a functional of the stress distribution of the body. If nonlocal strain is replaced by its local counterpart in $(9)_1$, however, it then becomes reasonable to assume that under some mild conditions the stress–strain relation is invertible. This makes the case of *restricted nonlocality* in which total strain is forced to remain local of particular interest. Note, in view of (2) that, if one selects as the attenuation function w^e a Dirac delta function, both (3) and (4) reduce to

$$\tilde{w}^e = \delta, \tag{17}$$

and hence, in view of $(8)_1$, that

$$\langle \boldsymbol{\varepsilon} \rangle = \boldsymbol{\varepsilon}, \tag{18}$$

and that the constitutive assumption (9) becomes

$$\left.\begin{aligned} \boldsymbol{\sigma} &= \tilde{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \langle \boldsymbol{\varepsilon}^p \rangle, \langle \kappa \rangle), \\ \psi &= \tilde{\psi}(\boldsymbol{\varepsilon}, \langle \boldsymbol{\varepsilon}^p \rangle, \langle \kappa \rangle), \end{aligned}\right\} \tag{19}$$

whereas, in view of (5), that (14) becomes

$$\boldsymbol{\sigma} = \frac{\partial \tilde{\psi}}{\partial \boldsymbol{\varepsilon}}. \tag{20}$$

Clearly the form of restricted nonlocality appearing in $(19)_1$ legitimates the assumption that, within a certain range (defined by a prescribed yield criterion), $\tilde{\boldsymbol{\sigma}}$ possesses an inverse of the following form:

$$\boldsymbol{\varepsilon} = \tilde{\boldsymbol{\varepsilon}}(\boldsymbol{\sigma} \langle \boldsymbol{\varepsilon}^p \rangle, \langle \kappa \rangle). \tag{21}$$

Apparently, we have made a decision regarding the level of generality here, since the choice of (17) restricts the theory considerably.

2.2.2. *Principle of maximum dissipation.* To illustrate the basic problems associated with the strain softening bar, it is sufficient to consider the simple case in which

$$\tilde{\psi} = \tfrac{1}{2}(\boldsymbol{\varepsilon}-\langle\boldsymbol{\varepsilon}^p\rangle\cdot\mathscr{L}(\boldsymbol{\varepsilon}-\langle\boldsymbol{\varepsilon}^p\rangle)+\tfrac{1}{2}H\langle\kappa\rangle^2, \tag{22}$$

where $\mathscr{L}$ is the tensor of elastic constants and H the plastic modulus (less than zero if the material exhibits softening behaviour). Note that (22) is in agreement with the general assumption $(19)_2$ and that (20) then becomes

$$\boldsymbol{\sigma} = \mathscr{L}(\boldsymbol{\varepsilon}-\langle\boldsymbol{\varepsilon}^p\rangle) = -\frac{\partial\tilde{\psi}}{\partial\langle\boldsymbol{\varepsilon}^p\rangle}, \tag{23}$$

where the second equality follows from (22) and $(23)_1$. Using (23), the plastic dissipation (16) takes the form

$$\begin{aligned}\mathbb{D}^p &= \int_B [\overline{\{\boldsymbol{\sigma}\}}_p\cdot\dot{\boldsymbol{\varepsilon}}^p+\overline{\{q\}}_h\dot{\kappa}]\,\mathrm{d}V\\ &= \int_B [\boldsymbol{\sigma}\cdot\langle\dot{\boldsymbol{\varepsilon}}^p\rangle+q\langle\dot{\kappa}\rangle]\,\mathrm{d}V,\end{aligned} \tag{24}$$

with q defined as

$$q = -\frac{\partial\tilde{\psi}}{\partial\langle\kappa\rangle} = -H\langle\kappa\rangle. \tag{25}$$

Apparently $\boldsymbol{\sigma}$ and q are generalized forces conjugated to the nonlocal variables $\langle\boldsymbol{\varepsilon}^p\rangle$ and $\langle\kappa\rangle$, respectively.

If

$$D^p = \overline{\{\boldsymbol{\sigma}\}}_p\cdot\dot{\boldsymbol{\varepsilon}}^p+\overline{\{q\}}_h\dot{\kappa} \tag{26}$$

is defined as being a measure of specific plastic dissipation (dissipation at unit volume), one can treat

$$D^p \geqslant 0 \tag{27}$$

as a localized form of the dissipation inequality, one which corresponds to a vanishing nonlocal residual (Edelen (1976)). Since the inequality (27) implies (15), this is a sufficient (although, of course, not a necessary) condition for the second law (10) to hold true.

If one assumes that (27) is valid for every set $\{\dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}\}$ at a given elastic–plastic state $\langle\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, \kappa\}$, the flow rules of nonlocal associated plasticity can be derived by invoking a generalized form of the principle of maximum dissipation in classical plasticity (Hill (1948)).† Recall that a normality condition for plastic strain and an associated condition of the convexity of the yield surface are the only restrictions placed on the constitutive equations by the principle in its classical form. If the principle is invoked in a more restrictive sense, however, further restrictions are placed on the constitutive functions, as indicated by the following.‡ For a prescribed yield function f, define a set A of admissible states

$$A = \{(\boldsymbol{\sigma}, q)\,|\,f(\boldsymbol{\sigma}, q) \leqslant 0\}, \tag{28}$$

and require for any given $\{\dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}\}$ that

† By Hill credited to von Mises (1928).
‡ In the case of local plasticity, a similar argument is used by Simo (1988).

$$D^p(\boldsymbol{\sigma}, q; \dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}) \geqslant D^p(\hat{\boldsymbol{\sigma}}, \hat{q}; \dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}) \tag{29}$$

for every possible state $\{\hat{\boldsymbol{\sigma}}, \hat{q}\} \in A$. Thus the principle claims that the *actual* state is the one for which the specific plastic dissipation adopts its maximum. Proceed by constructing a Lagrangian

$$L^p(\boldsymbol{\sigma}, q, \dot{\gamma}) = -D^p(\boldsymbol{\sigma}, q; \dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}) + \dot{\gamma} f(\boldsymbol{\sigma}, q), \tag{30}$$

where $\dot{\gamma} \geqslant 0$ is a Lagrangian multiplier. The condition (29) is now enforced by solving the associated constrained minimization problem implied by (30), i.e.,

$$\frac{\partial L^p}{\partial \boldsymbol{\sigma}} = \mathbf{0}, \quad \frac{\partial L^p}{\partial q} = 0, \quad \dot{\gamma} \geqslant 0, \quad \dot{\gamma} f(\boldsymbol{\sigma}, q) = 0. \tag{31}$$

From (5), (6), (26), (28) (30) and (31) it then follows that

$$\left.\begin{aligned} \dot{\boldsymbol{\varepsilon}}^p &= \frac{1}{\beta^p}\dot{\gamma}\frac{\partial f}{\partial \boldsymbol{\sigma}}, \\ \dot{\kappa} &= \frac{1}{\beta^h}\dot{\gamma}\frac{\partial f}{\partial q}, \end{aligned}\right\} \tag{32}$$

$$\dot{\gamma} \geqslant 0, \quad f(\boldsymbol{\sigma}, q) \leqslant 0, \quad \dot{\gamma} f(\boldsymbol{\sigma}, q) = 0, \tag{33}$$

where the two equations displayed in (32) are the flow rules of (nonlocal) associated plasticity, whereas (33) represents the corresponding loading/unloading conditions (in Kuhn–Tucker form).

If the attenuation functions w^p and w^h coincide, then $\beta^p = \beta^h = \beta$. This allows the functions to be incorporated into $\dot{\gamma}$ (since $\beta > 0$). Hence, (32) and (33) can be replaced by†

$$\left.\begin{aligned} \dot{\boldsymbol{\varepsilon}}^p &= \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}, \\ \dot{\kappa} &= \dot{\lambda}\frac{\partial f}{\partial q}, \end{aligned}\right\} \tag{34}$$

$$\dot{\lambda} \geqslant 0, \quad f(\boldsymbol{\sigma}, q) \leqslant 0, \quad \dot{\lambda} f(\boldsymbol{\sigma}, q) = 0, \tag{35}$$

where $\dot{\lambda}$ is defined as

$$\dot{\lambda} = \beta\dot{\gamma}. \tag{36}$$

Remarks 2.3. The nonlocal character of the theory should be emphasized. Although (34) along with the requirement (35) appear in a seemingly local form, both (19) and (25) clearly reveal the nonlocal status of the flow rules and the loading/unloading conditions. The restriction of the nonlocal formulation to local theory is obtained by simply discarding the $\langle\rangle$ symbols and the averaging operator $\overline{\{\ \}}$ wherever they occur. In the local case, the convexity of the yield surface is a consequence of the principle of maximum dissipation, as can be shown by the following simple argument: choose $\hat{q} = q$ in (29) and conclude, with the aid of (26), that

† Alternatively, boundary effects can be assumed to be negligible and β can be set equal to unity (cf. Section 3.2).

$$(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}})\cdot\dot{\boldsymbol{\varepsilon}}^p \geqslant 0, \tag{37}$$

which, in view of either $(32)_1$ or $(34)_1$, implies convexity. In the *nonlocal* case, however, the corresponding argument is not valid: the nonlocal counterpart of (37) reads

$$\overline{\{\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}\}}_p\cdot\dot{\boldsymbol{\varepsilon}}^p \geqslant 0, \tag{38}$$

and, since (37) does not in general follow from (38), convexity is not implied by the generalized principle of maximum dissipation expressed by (28) and (29).

Recall that in a corresponding local theory the yield surface is stationary when $\dot{\boldsymbol{\varepsilon}}^p(\mathbf{x}) = \mathbf{0}$ and $\dot{\kappa}(\mathbf{x}) = 0$ (i.e., for fixed values of $\boldsymbol{\varepsilon}^p$ and κ at $\mathbf{x}$). From either (32) and (33) or (34) and (35) it follows then that in an elastic state the yield surface is stationary. The proposed nonlocal formulation does not afford any such simple geometrical interpretation since, whereas in an elastic state $\dot{\boldsymbol{\varepsilon}}^p(\mathbf{x})$ and $\dot{\kappa}(\mathbf{x})$ still vanish (since $\dot{\gamma} = 0$), $\langle\dot{\boldsymbol{\varepsilon}}^p\rangle(\mathbf{x})$ and $\langle\dot{\kappa}\rangle(\mathbf{x})$ in general do not. Hence, in view of (25), the yield surface is not necessarily stationary at $\mathbf{x}$ but may change due to plastic deformation at other parts of the body (i.e., $\dot{\boldsymbol{\varepsilon}}^p(\mathbf{z}) \neq \mathbf{0}$, $\dot{\kappa}(\mathbf{z}) \neq 0$ in some finite region). □

3. STRAIN SOFTENING BAR

The constitutive three-dimensional formulation presented in the previous section is rather general, restricted of course by the choice (22) of the Helmholtz free energy, but without restrictions being placed on the yield function f (except for the usual smoothness conditions). In fact, a suitable choice of f provides for the nonlocal formulation of several well-known models of rate-independent plasticity. Here, attention will be confined to a one-dimensional formulation of the general theory involving the choice of a simple yield function corresponding to linear strain softening. From this it follows that the problem of localization in the strain softening bar is easy to solve analytically.

Specifically, assume that the bar has length $L = 0.1$ m, Young's modulus $E = 20{,}000$ MPa (constant along the entire bar), and the strain hardening modulus $H = -0.05E$. The initial yield stress is $\sigma_y = 2$ MPa everywhere, except for some small region in which it is reduced by a certain given amount. The bar is prescribed to be loaded in uniaxial tension.

3.1. *Elastic–plastic equations*

A one-dimensional formulation of the Helmholtz free energy which complies with (22) is chosen:

$$\tilde{\psi} = \tfrac{1}{2}E(\varepsilon-\langle\varepsilon^p\rangle)^2+\tfrac{1}{2}H\langle\kappa\rangle^2, \quad E+H > 0. \tag{39}$$

According to (23) and (39)

$$\sigma = E(\varepsilon-\langle\varepsilon^p\rangle). \tag{40}$$

A yield function corresponding to linear strain softening is defined by

$$f = \sigma-\sigma_y+q, \quad \sigma > 0, \tag{41}$$

where it should be recalled that $q = -H\langle\kappa\rangle$, in agreement with the definition given by (25). If $H \neq 0$, it then follows from (32) and (41) that

$$\left.\begin{aligned}\dot{\varepsilon}^p &= \frac{1}{\beta^p}\dot{\gamma},\\ \dot{\kappa} &= \frac{1}{\beta^h}\dot{\gamma},\end{aligned}\right\} \tag{42}$$

where $\dot{\gamma}$ is subjected to the requirement (33).

3.2. *Localized zone*

If strain softening in the bar is initiated at a point x_0 where the initial yield stress is reduced, say, to $\sigma_y = 1.4$ MPa, one can assume that

$$\dot{\gamma}(x) = \dot{B}(x)\delta(x-x_0), \tag{43}$$

where $\delta(x)$ is a Dirac delta function which satisfies

$$\int_{-L/2}^{L/2} \delta(x-x_0)\,\mathrm{d}x = 1, \tag{44}$$

and where B is some as yet unknown function. Hence, according to (42) and (43),

$$\left.\begin{aligned}\dot{\varepsilon}^p(x) &= \frac{\dot{B}(x)}{\beta^p(x)}\delta(x-x_0),\\ \dot{\kappa}(x) &= \frac{\dot{B}(x)}{\beta^h(x)}\delta(x-x_0).\end{aligned}\right\} \tag{45}$$

It then follows from (8) and (45) that the nonlocal plastic strain rate becomes

$$\langle\dot{\varepsilon}^p\rangle(x) = \int_{-L/2}^{L/2} \tilde{w}^p(z,x)\frac{\dot{B}(z)}{\beta^p(z)}\delta(z-x_0)\,\mathrm{d}z = \frac{\dot{B}(x_0)}{\beta^p(x_0)}\tilde{w}^p(x_0,x), \tag{46}$$

and similarly that

$$\langle\dot{\kappa}\rangle(x) = \frac{\dot{B}(x_0)}{\beta^h(x_0)}\tilde{w}^h(x_0,x). \tag{47}$$

In view of (35) and (43), the consistency condition of plasticity implies that $\dot{f} = 0$ at x_0. It then follows, by use of (41) and (47), that

$$\dot{\sigma} = H\langle\dot{\kappa}\rangle(x_0) = H\frac{\dot{B}(x_0)}{\beta^h(x_0)}\tilde{w}^h(x_0,x_0) \tag{48}$$

and hence, from (3) (recall that by assumption $w(0) = 1$), that

$$\dot{B}(x_0) = V_h(x_0)\beta^h(x_0)\frac{\dot{\sigma}}{H}. \tag{49}$$

Using (40), (46) and (49), one can conclude that the relationship between strain rate and stress rate becomes

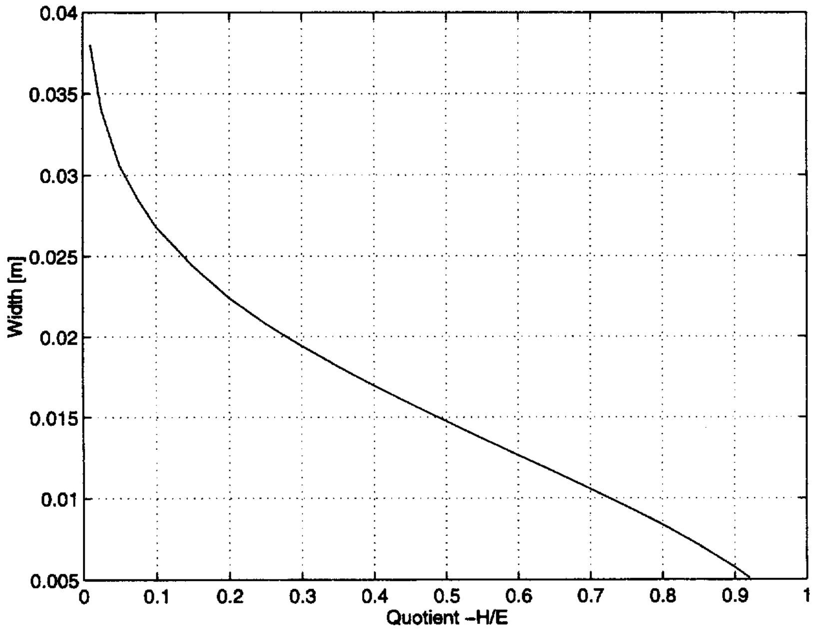


Fig. 1. Width of localized zone (Gaussian attenuation function).

$$\dot{\varepsilon}(x) = \dot{\sigma}\left[\frac{1}{E} + \frac{1}{H} V_h(x_0) \frac{\beta^h(x_0)}{\beta^p(x_0)} \tilde{w}^p(x_0, x)\right], \tag{50}$$

valid for all $x \in [-L/2, L/2]$.

To simplify, assume that† $w^p = w^h = w$ and that strain softening is initiated at the centre of the bar, i.e., at $x_0 = 0$. The width b of the localized zone is then

$$b = 2x_b, \tag{51}$$

where x_b is a solution to the equation $\dot{\varepsilon} = 0$, i.e.,

$$V(0)\tilde{w}(0, x) + \frac{H}{E} = 0, \tag{52}$$

according to (50). Specifically, choosing a Gaussian attenuation function of the form

$$w = w(|z-x|) = e^{-\pi(z-x)^2/l^2}, \tag{53}$$

it can be seen that (52) explicitly reads

$$\left.\begin{aligned} \frac{V(0)}{V(x)} e^{-\pi x^2/l^2} + \frac{H}{E} = 0, \\ V(x) = \int_{-L/2}^{L/2} e^{-\pi(z-x)^2/l^2}\,\mathrm{d}z, \end{aligned}\right\} \tag{54}$$

in accordance with (2) and (3). The solution of (54) is shown graphically for $l = 0.0157$ m in Fig. 1. From $(54)_2$ one finds that

† Note that in this case (50) yields $\dot{\sigma} = EH/(E+H)\,\dot{\varepsilon}(x_0)$, a relationship which, during loading, also follows directly from (40)–(42) if $\beta^p = \beta^h$.

$$\lim_{L\to\infty} V(0) = l. \tag{55}$$

Hence, if

$$l/L \ll 1, \tag{56}$$

then $V(x) = V(0)$ almost everywhere (except for a narrow zone close to the boundary), and (54) has approximately a closed form solution

$$x_b = \pm l\left(\frac{1}{\pi}\ln\frac{E}{-H}\right)^{1/2}. \tag{57}$$

Since $l/L = 0.157$ here, (57) is an acceptable solution, as can easily be checked by a comparison with the graph in Fig. 1. In particular, solving (54) gives $b = 0.0314$ m ($\approx 2l$) for $-H/E = 0.05$, whereas (57) gives $b = 0.0307$ m.

The integration of (50) ($w^p = w^h = w$) provides the total displacement rate of the bar

$$\begin{aligned}\Delta\dot{u} &= \dot{u}\left(\frac{L}{2}\right)-\dot{u}\left(-\frac{L}{2}\right)\\ &= \dot{\sigma}\int_{-L/2}^{L/2}\left[\frac{1}{E}+\frac{1}{H}V(x_0)\tilde{w}(x_0,x)\right]\mathrm{d}x = \dot{\sigma}\left(\frac{L}{E}+\frac{l_{ch}(x_0)}{H}\right),\end{aligned} \tag{58}$$

where (6) has also been used and where

$$l_{ch}(x_0) = V(x_0)\beta(x_0) \tag{59}$$

is some characteristic length which at a given point and for a given length of the bar depends on the character of the attenuation function alone. Since the representative volume $V(x)$ only deviates appreciably from $V(0)$ at the boundary, note in particular that

$$\beta(0) \approx \frac{1}{V(0)}\int_{-L/2}^{L/2} w(0,x)\,\mathrm{d}x = 1 \tag{60}$$

for rapidly decaying attenuation functions ($l/L \ll 1$). For the Gaussian distribution function (53), it turns out that (60) is in fact accurately satisfied if $l/L < 0.3$, as can be seen in Fig. 2a. In this specific case (in which $x_0 = 0$), (59) is replaced by

$$l_{ch} = l_{ch}(0) \approx V(0), \tag{61}$$

and hence, due to (55),

$$l_{ch} \approx l, \tag{62}$$

independent of the length of the bar.

One can also observe in Fig. 2a that the integral of $\tilde{w}(0,x)$ deviates from unit value by only a small amount. Accordingly, it can be argued that (60) is valid even if (56) is not valid. The case in which x_0 is an arbitrary point along the bar is illustrated in Fig. 2b, in which the function $l_{ch}(x_0)/V(x_0) = \int \tilde{w}(x_0,x)\,\mathrm{d}x$ is shown for five different values of the quotient l/L. For small values of l/L (rapidly decaying Gaussian attenuation function), one can observe that the integral of $\tilde{w}(x_0,x)$ only deviates from unity in small regions close to the boundaries.

The stress–displacement relation is obtained from (58)

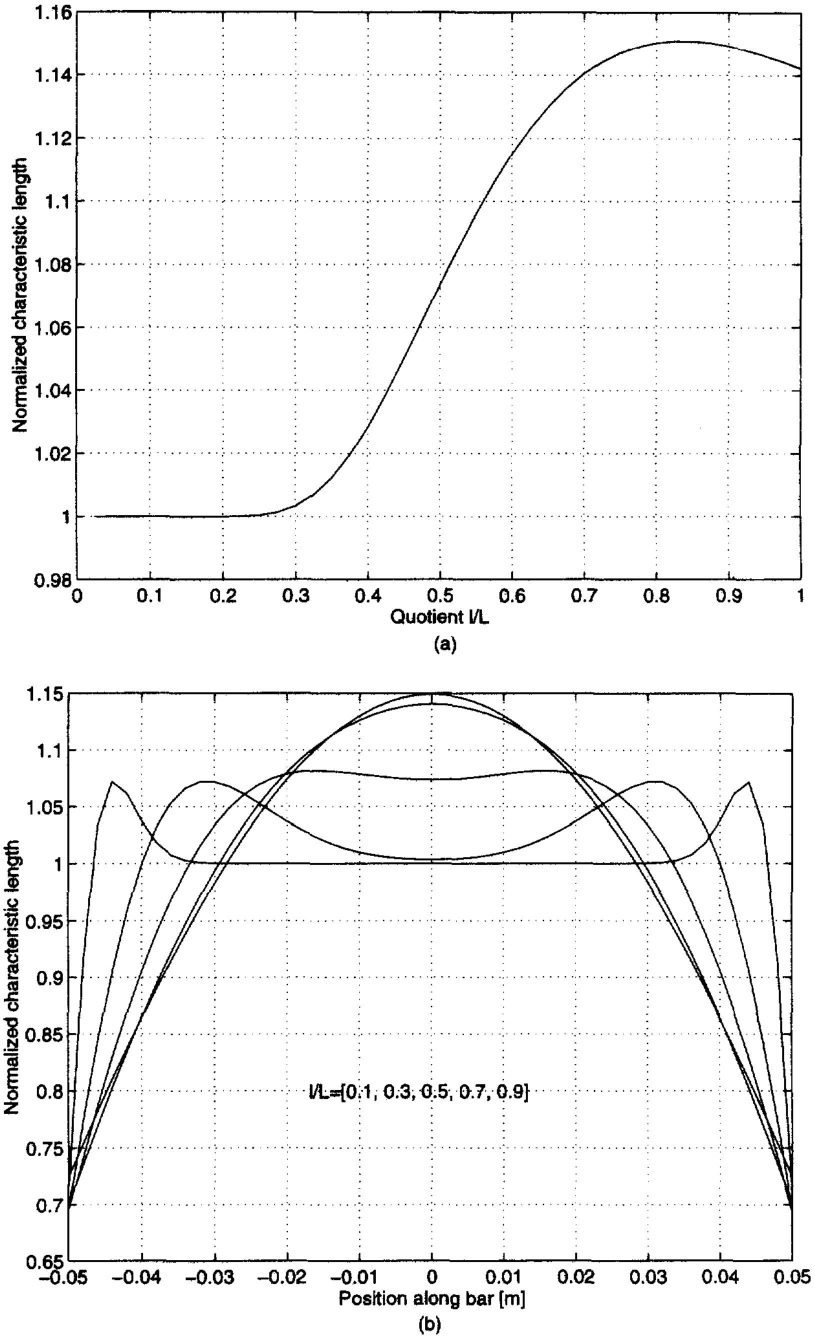
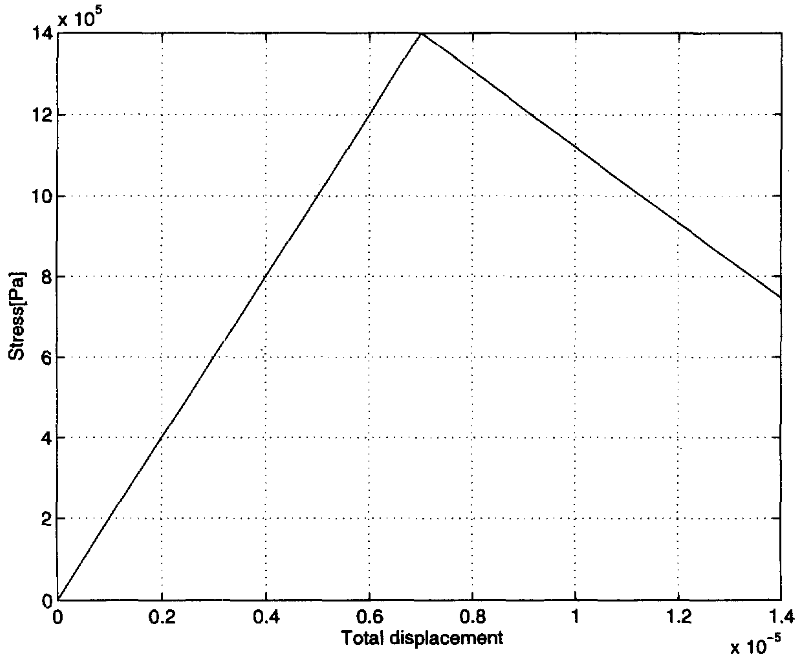


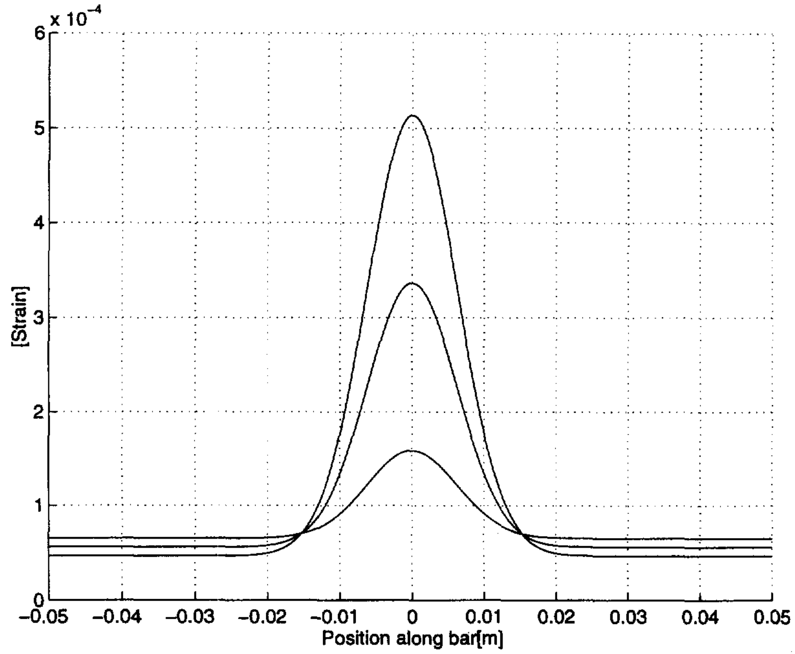
Fig. 2. Characteristic length l_{ch} with respect to Gaussian attenuation function. (a) The function $l_{ch}/V(0) = \int \bar{w}(0, x)\,\mathrm{d}x$. (b) The function $l_{ch}/V(x_0) = \int \bar{w}(x_0, x)\,\mathrm{d}x$ for five different values of quotient l/L.

$$\sigma = \begin{cases} \dfrac{E}{L}\Delta u, & 0 < \Delta u < \dfrac{L\sigma_y}{E}, \\ \sigma_y + \dfrac{E}{L\left(1 + \dfrac{E}{H}\dfrac{l_{ch}(x_0)}{L}\right)}\left(\Delta u - \dfrac{L\sigma_y}{E}\right), & \Delta u > \dfrac{L\sigma_y}{E}, \end{cases} \tag{63}$$

and is shown in Fig. 3a for $l = 0.0157$ m and $x_0 = 0$ (σ_y then being the reduced initial yield stress at the centre of the bar).



(a)



(b)

Fig. 3. Stress and strain (Gaussian attenuation function). (a) Stress versus total displacement ($l/L = 0.157$). (b) Strain distribution along the bar for three different values of total displacement ($l/L = 0.157$).

The strain distribution for different values of the total displacement of the bar are obtained from (50) and (51). The result (for $x_0 = 0$) is shown in Fig. 3b for values of stress corresponding to $\Delta u = (0.8, 1.0, 1.2) \times 10^{-5}$ m.

For $l/L = 0.157$ the width of the localized zone has been calculated by numerically solving the integral eqn (54) for different values of $-H/E$. In particular $b = 0.0314$ m ($\approx 2l$) if $-H/E = 0.05$, which is verified by the strain distribution shown in Fig. 3b. As expected, the width of the zone remains constant when the loading is increased.

Remarks 3.1. The analysis above highlights the difference between permanent deformation and plastic flow in general nonlocal plasticity. Recall that the stress point never reaches the yield surface except for the single point at x_0. Yet it appears to clearly be the case that permanent deformation remains within the entire localized zone if the stress is relaxed to zero. However, in the special case of restricted nonlocality in which $w^p = \delta \neq w^h$ (and hence $V_p(x) = \beta^p(x) = 1$ as seen from (2), (4) and (6)), the plastic strain remains local and (50) becomes

$$\dot{\varepsilon}(x) = \dot{\sigma}\left(\frac{1}{E} + \frac{1}{H} V_h(x_0)\beta^h(x_0)\delta(x-x_0)\right). \tag{64}$$

Thus the strain is localized in a zone of vanishing size at x_0, whereas the total displacement (58) remains unaltered, being discontinuous at x_0, with a jump in magnitude of $(\dot{\sigma}/H)l_{ch}$ (since l_{ch} does not depend on w^p, as can be easily seen from (50), (58) and (59)). It may then be argued that use of a nonlocal model in which the attenuation function w^p is chosen as a delta function supports the concept of a cohesive crack model, as has been discussed recently by Planas *et al.* (1993, 1994). □

3.3. *Identification of nonlocal characteristic length*

In the case discussed in Section 3.2 in which $w^p = w^h$ (cf. (52) and (54)), the width of the localized zone (for given L) depends specifically on the quotient H/E and the constitutive parameter l (referred to by some authors as simply *the nonlocal characteristic length* of the material). It is evidently of considerable importance to relate this parameter to the physical properties of the material. Taking account of the dissipation in the bar provides an interpretation of l (associated with the attenuation function w^h), as will be demonstrated below.

The total dissipation in the bar is obtained from either $(24)_1$ or $(24)_2$. Using the second alternative, it follows with the aid of (46), (47) and (25) that

$$\begin{aligned}\mathbb{D}^p &= \dot{B}(x_0)\int_{-L/2}^{L/2}\left\{\sigma\frac{\tilde{w}^p(x_0,x)}{\beta^p(x_0)} - H\langle\kappa\rangle\frac{\tilde{w}^h(x_0,x)}{\beta^h(x_0)}\right\}\mathrm{d}x \\ &= V_h(x_0)\beta^h(x_0)\frac{\dot{\sigma}\sigma}{H} - V_h(x_0)\dot{\sigma}\int_{-L/2}^{L/2}\langle\kappa\rangle\tilde{w}^h(x_0,x)\,\mathrm{d}x,\end{aligned} \tag{65}$$

where the second equality is due to (6), (49) and the fact that σ does not depend on x. In view of (49), the integration of (47) (with properly chosen initial condition) gives

$$\langle\kappa\rangle(x) = V_h(x_0)\tilde{w}^h(x_0,x)\frac{(\sigma-\sigma_y)}{H}, \tag{66}$$

and hence, using definition (59),

$$\mathbb{D}^p = l_{ch}\frac{\dot{\sigma}\sigma}{H} - l^*_{ch}\frac{\dot{\sigma}(\sigma-\sigma_y)}{H}, \tag{67}$$

where

$$l^*_{ch} = V^2(x_0)\int_{-L/2}^{L/2}(\tilde{w}(x_0,x))^2\,\mathrm{d}x \tag{68}$$

is another characteristic length, and where $\tilde{w}$ refers to the strain hardening function κ (the superscript h being suppressed).

Remarks 3.2. It should be noted that (67) implies that $\mathbb{D}^p$ does not depend on w^p. Hence, the dissipation remains nonzero, even when $w^p = \delta$.

It should also be noted that the dissipation inequality (15) places no restrictions upon the attenuation functions since each term in the right hand side of (67) is non-negative. Furthermore, the condition $(33)_2$ implies that

$$\mathbb{D}^p \leqslant l_{ch} \frac{\dot{\sigma}\sigma_y}{H}, \tag{69}$$

as can be seen from (65), (41) and (6)):

$$\begin{aligned}\mathbb{D}^p &= \dot{B}(x_0)\left\{\sigma + \frac{1}{\beta(x_0)}\int_{-L/2}^{L/2}(f-(\sigma-\sigma_y)\tilde{w}(x_0,x))\,\mathrm{d}x\right\} \\ &= \dot{B}(x_0)\left\{\sigma_y + \frac{1}{\beta(x_0)}\int_{-L/2}^{L/2} f\tilde{w}(x_0,x)\,\mathrm{d}x\right\} \leqslant \dot{B}(x_0)\sigma_y,\end{aligned} \tag{70}$$

which, in view of (49), is (69).

In addition, it can be noted on the basis of (67) and (69) that the characteristic lengths l_{ch} and l_{ch}^* must fulfill

$$l_{ch}^* \leqslant l_{ch}, \tag{71}$$

an equality trivially satisfied for an attenuation function of the Gaussian type (for which $\tilde{w}(x_0, x)$ attains its maximum value at $x = x_0$):

$$l_{ch}^* \leqslant V^2(x_0)\tilde{w}(x_0,x_0)\int_{-L/2}^{L/2}\tilde{w}(x_0,x)\,\mathrm{d}x = l_{ch}, \tag{72}$$

where advantage has been taken of the definitions of l_{ch} and l_{ch}^* ((59) and (68), respectively) and of (3) together with the fact that $w(0) = 1$. □

At failure (complete separation of surfaces), the total amount of dissipation is given by $2G_c$, where G_c is the fracture energy per unit area for the specimen in question. Hence, since $\sigma = 0$ and $\dot{\sigma} = -\sigma_y$ at failure,

$$\frac{\sigma_y^2}{-H} l_{ch}^* = 2G_c, \tag{73}$$

which for an attenuation function of type (53) is an integral equation for the parameter l (for given values of x_0 and L).

If one assumes that G_c represents a true material property (neglecting any size effect) (73) can be written in the form†

$$l_{ch}^* = \frac{-H}{E}\lambda_c, \tag{74}$$

where

$$\lambda_c = \frac{2G_c E}{\sigma_y^2} \tag{75}$$

is a characteristic length of the material, commonly used by researchers in the field of

† It is tacitly assumed that neither σ_y nor E depends on the size of the specimen.

fracture mechanics (see Hillerborg *et al.* (1976)). The parameter l can then be explicitly determined from (74). Choosing $x_0 = 0$ and assuming that (56) is valid, it is found, in view of (53), that approximately

$$l^*_{ch} = \frac{l}{\sqrt{2}}, \tag{76}$$

and hence that

$$l = \sqrt{2}\frac{-H}{E}\lambda_c. \tag{77}$$

Remarks 3.3. For $l = 0.0157$ m, $-H/E = 0.05$ (cf. Fig. 1), it can be seen from (77) that $\lambda_c \approx 0.22$ m, which appears to be a typical value for mortar. Stable experiments on strain softening specimens (displacement controlled measurements) require that $\lambda_c > L$ (see e.g., Hillerborg (1989)). Since $L = 0.1$ m here, this stability condition is fulfilled.

It should also be noted that (71) and (74) together imply that $l \geqslant (-H/E)\lambda_c$ for *any* attenuation function which satisfies (62) (verified above by (77) for a Gaussian attenuation function). □

4. DISCUSSION AND CONCLUSIONS

To be discussed below are the question of the uniqueness of the equilibrium solution of the elastic–plastic equations (Section 4.1), and the nonequivalence between nonlocal plasticity and gradient plasticity (Section 4.2). Various concluding remarks are also provided (Section 4.3).

4.1. *Uniqueness of equilibrium solution*

In Section 3.2, strain softening was assumed to be initiated at a single point x_0, forcing the flow rules to satisfy (45). Here it will be shown that $(45)_2$ in fact represents the only nontrivial equilibrium solution of the elastic–plastic equations—at least for a particular class of attenuation functions.

It follows from (41) and the consistency condition of plasticity that during plastic loading

$$\dot{\sigma} = H\int_{-L/2}^{L/2} \tilde{w}(z,x)\dot{\kappa}(z)\,\mathrm{d}z, \tag{78}$$

where $\tilde{w}(z,x)$ refers to the strain hardening function κ (the superscript h being omitted). Equilibrium requires that $\partial\sigma/\partial x = 0$ and hence that†

$$\frac{\partial\dot{\sigma}}{\partial x} = 0. \tag{79}$$

One trivial solution of (79) which complies with (78) is

$$\dot{\kappa} = \frac{\dot{\sigma}}{H}, \tag{80}$$

which represents uniform hardening of the bar. Another solution is the one discussed in Section 3.2, i.e.,

† For a locally mass closed body in which long range gravitational effects are negligible, the equations of linear and rotational momentum coincide with those of local theory (Edelen (1976)).

$$\dot{\kappa}(z) = \frac{\dot{\sigma}}{H} \frac{\delta(z-x)}{\tilde{w}(x,x)} = \frac{\dot{\sigma}}{H} V(x)\delta(z-x), \tag{81}$$

which represents softening at a single point x (the second equality being due to (3)). It can be easily shown that (80) and (81) are in fact for a wide class of attenuation functions the only solutions to (79). If one ignores the trivial solution (80) and further assumes that

$$\dot{\kappa}(x) > 0 \quad \text{for} \quad x \in [x_1, x_2] \tag{82}$$

in some finite region, (79) should then require that

$$\int_{x_1}^{x_2} \tilde{w}'(z,x)\dot{\kappa}(z)\,\mathrm{d}z = 0, \tag{83}$$

where $\tilde{w}'$ is the derivative of the function $\tilde{w}$ with respect to x. Since it is easy to show that this is not possible for attenuation functions of the type (53), (81) is the only nontrivial solution of (78) and (79); cf. Planas *et al.* (1994).†

4.2. *Comparison with gradient theory*

Since several authors characterize gradient materials as *nonlocal*, it is important to note that—according to the terminology of Noll (1958) and to that adopted in the present paper—materials with gradient effects are *not* nonlocal but belong to a certain class of *non-simple* materials. The fact that functionals (present in nonlocal theories) may be formally approximated by the use of Taylor expansions does not justify the idea (adopted by some authors) that constitutive relationships for gradient materials can be derived from corresponding constitutive equations for nonlocal materials.

In fact, already in the results obtained in the previous section, the nonequivalence between nonlocal plasticity and gradient plasticity is manifest. To see this clearly, note that (55) allows one to assume that $V(x) \approx l$ for a fixed L and for a sufficiently small l, provided x is not too close to the boundary. Within the limits of this approximation one can express the nonlocal strain hardening function $(8)_3$ (after a change of integration variable) in the form

$$\langle\kappa\rangle(x) = \frac{1}{l}\int_{-\infty}^{\infty} w(s)\kappa(x+s)\,\mathrm{d}s, \tag{84}$$

with

$$\int_{-\infty}^{\infty} w(s)\,\mathrm{d}s = l. \tag{85}$$

The consistency with local theory is evident since

$$\lim_{l\to 0}\langle\kappa\rangle(x) = \kappa(x), \tag{86}$$

which is easy to show for attenuation functions of type (53) if κ is assumed to be continuous. In particular, note that the relationship

† In a slightly different context (corresponding to the special case of restricted nonlocality with $w^p = \delta$, as discussed in Section 3.2), Planas and coworkers have proved, for a large class of attenuation functions, in particular including (53), the nonexistence of an inelastic distribution over a finite interval.

$$w(s) = w(-s) \tag{87}$$

is valid for these functions.†

Through expanding $\kappa(x+s)$ in a Taylor series at x and using (85) and (87), it is then possible to write (84) in the form

$$\begin{aligned} \langle\kappa\rangle(x) &= \frac{1}{l}\int_{-\infty}^{\infty} w(s)\left(\kappa(x)+s\kappa'(x)+\frac{s^2}{2}\kappa''(x)+\ldots\right)\mathrm{d}s \\ &= \kappa(x)+\alpha_1 l^2\kappa''(x)+\alpha_2 l^4\kappa^{\mathrm{IV}}(x)+\ldots, \end{aligned} \tag{88}$$

where α_1 and α_2 are constants (independent of l). For example, it is found that the value of α_1 is $0.25/\pi$ if the Gaussian attenuation function (53) is employed.

It can be anticipated that a gradient plasticity model based on a yield function that depends both on the second derivative of the strain hardening function and on the function itself, can be derived to any desired degree of accuracy from (88) by simply neglecting all terms of higher order (a sufficiently small l being chosen and its being tacitly assumed that the subsequent derivatives of κ remain bounded). In general, however, such an argument is not a rigorous one, as discussed by Huerta and Pijaudier-Cabot (1994). Here, the uniqueness of the equilibrium solution (81) provides another proof of the nonequivalence of nonlocal methods and the corresponding gradient ones. To demonstrate this, one can examine the gradient model derived by de Borst and Mühlhaus (1992), which is based on the following constitutive equations:

$$\left.\begin{aligned} \sigma &= E(\varepsilon-\varepsilon^p), \\ f = \sigma-\sigma_y-H\bar{\kappa}, &\quad \bar{\kappa} = \kappa+l_G\kappa'', \\ \dot{\varepsilon}^p &= \dot{\kappa} = \dot{\gamma}, \end{aligned}\right\} \tag{89}$$

where l_G is some internal length. This set of equations should be compared with the nonlocal formulation (39)–(42). Hence, one needs to select $\langle\varepsilon^p\rangle = \varepsilon^p$, i.e., the special form of restricted nonlocality in which $w^p = \delta$ (recall that $\beta^h \approx 1$ by assumption). Furthermore, in view of (88), the internal length l_G should be chosen as

$$l_G = \sqrt{\alpha_1}\,l. \tag{90}$$

The equilibrium solution of (89) is represented by the plastic strain distribution

$$\dot{\varepsilon}^p(x) = \dot{A}\cos(x/l_G)+\frac{\dot{\sigma}}{H}, \tag{91}$$

($\dot{A}$ being independent of x), whereas the width of the localized zone is given by

$$b = 2\pi l_G. \tag{92}$$

However, as shown in Section 4.1, the corresponding nonlocal solution has the form (81), which means that for the class of attenuation functions in question a plastic strain distribution over a finite interval is an impossibility. Hence the two models are not equivalent and gradient relations of the type discussed here cannot, in general, be derived from nonlocal theory.

† It is not necessary to restrict the choice of attenuation functions to (53), since the arguments to be discussed remain valid for a wide class of integrable attenuation functions that comply with (84)–(87) above.

4.3. *Concluding remarks*

The nonlocal approach provides in a natural way for the introduction of a set of characteristic lengths, these being the major parameters that control the development of the localized zone, as has been demonstrated by analysis of the strain softening bar. The analytical solution implies that the width of the localized zone depends on the length of the bar, although this size effect is negligible if the quotient l/L is small, l being a nonlocal characteristic length and L the length of the bar. For a given type of attenuation function and for sufficiently small values of the quotient l/L, the width of the localized zone is determined entirely by the parameter l, Young's modulus and the strain softening modulus. It was found further that the analytical solution predicts a finite amount of total dissipation, despite plastic loading being confined to a region of vanishing size at the centre of the bar. Comparison with the total separation work at failure allows an integral equation to be derived which can be solved so as to obtain the nonlocal parameter l.

The appropriate choice of attenuation functions and the physical identification of the characteristic length (or lengths) of a nonlocal continuum are issues of great importance. In general, they need to be dealt with by a combination of micromechanical analysis and experimental investigation. Choosing attenuation functions properly is apparently not simply a matter of numerical convenience but can be of decisive importance for the localization process in general and for the final width of the localized zone in particular. Ultimately, only experimental investigation can provide the validation of the one choice or the other.

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